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USING FIRST INTEGRALS TO ESTIMATE LIMITING POSSIBILITIES OF OPTIMAL CONTROL SYSTEMS†

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In a continuation of previous investigations [1–4] devoted to the question of using first integrals in problems of the optimal control of dynamical systems, the optimal control problem for the motion of a multi-dimensional non-linear system over a given time interval with fixed endpoints for the phase trajectory is considered. The quality of the control is estimated by a functional in the form of a definite integral of a weighted sum of squares of the components of the controlling forces. In a number of cases [5–7] such a functional gives an estimate of energy losses during control, and the corresponding variational problem is called optimal for minimum energy loss. Based on the use of first integrals of the free equations of motion a method is developed to find the upper limit of the least necessary energy loss to displace the controlled non-linear dynamical system from a given initial phase state to a given final state in a given time. The efficiency of the method is illustrated by examples, including the solution of the problem of estimating the limiting possibilities of an energetically optimal control system for controlling the motion of an artificial satellite in a gravitationally attractive Newtonian central field.

IT IS well known [5–12] that the exact solution of optimal control problems is only possible very rarely, and only for special types of dynamical system. The search for approximate solutions of optimal problems also, as a rule, involves considerable difficulties [9–12]. Hence it is of interest to develop methods for estimating some or other of the properties of optimal control processes for dynamical systems without determining the exact solution of the variational problem itself.

1. Consider the multidimensional non-stationary non-linear dynamical system whose controlled motions are described by the equations

$$\begin{aligned} x^{\cdot} &= f(x,t)+B(x)\,u(x,t) \\ x &= (x_1,\ldots,x_n), \quad u = (u_1,\ldots,u_m) \end{aligned} \tag{1.1}$$

Here x is an n-dimensional vector of phase coordinates, u is an m-dimensional vector of controlling forces, and $f(x, t)$ and $B(x)$ are a given vector-function and matrix of dimensions $(n\times 1)$ and $(n\times m)$, respectively.

Suppose the states of the system at the initial time $t = 0$ and at the end of the control process at $t = T$ are given by the formulae

$$x(0) = x_0 \tag{1.2}$$

$$x(T) = x_T \tag{1.3}$$

where x_0 and x_T are given n-dimensional vectors, and T is a given positive number.

We shall call the pair of vector-functions $\{x(t),\ u(x,t)\}$ an admissible controlled process of the dynamical system (1.1) if these functions satisfy Eqs (1.1) for $t\in[0, T]$ and boundary conditions (1.2), (1.3).

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We shall assume that the set of admissible controlled processes of system (1.1) is not empty and denote it by Ω.

To estimate the quality of the controlled process we apply a functional of the form

$$J[x(\cdot),u(\cdot)] = \frac{1}{2}\int_0^T \sum_{j=1}^m \left[\frac{u_j(x,t)}{k_j}\right]^2 dt \tag{1.4}$$

where the k_j are given constants (weighting factors).

Suppose there exists an admissible controlled process $\{x_*(t),\ u_*(x,t)\}$ such that

$$\min_{\{x,u\}\in\Omega} J[x(\cdot),u(\cdot)] = J[x_*(\cdot),u_*(\cdot)]$$

We shall call a pair of functions $\{x_*(t),\ u_*(x,t)\}$ that satisfy this relation energetically optimal controlled processes, and the quantity $J[x_*(\cdot),\ u_*(\cdot)] = J_*$ the least necessary energy loss for displacing the dynamical system (1.1) from the given initial phase state (1.2) to the given final phase state (1.3) in time T.

The aim of the paper is to find the upper limit of the least necessary energy loss J_*.

There are various approaches to the search for this estimate. Firstly, one can, using known methods from the theory of optimal control, solve directly the variational problem of finding the minimum of the functional (1.4) with differential constraints (1.1) and boundary conditions (1.2), (1.3). After determining the extremals $\{x_*(t),\ u_*(x,t)\}$ the quantity J_* is calculated from formula (1.4). Secondly, an estimate of J_* can obviously be obtained by constructing or estimating the admissibility set of the dynamical system and solving the corresponding non-linear programming problem [13]. The advantage of one or other approach to obtaining an estimate of J_* is obviously governed by the specific form of system (1.1) and boundary conditions (1.2), (1.3).

Below a method is presented of estimating the least necessary energy loss to displace dynamical system (1.1) from state (1.2) to state (1.3). This method is based on the use of first integrals of the equations of free motion

$$x^{\cdot} = f(x,t) \tag{1.5}$$

We have the following theorem.

Theorem 1. Suppose that the function $w(x, t)$ is a first integral of Eqs (1.5) and is such that the following conditions are satisfied.

1. A functional of the form

$$G[x(\cdot)] = \int_0^T \sum_{j=1}^m k_j^2 \langle \nabla_x w(x,t),\ b_j(x)\rangle^2\, dt \tag{1.6}$$

is defined over the entire set of admissible controlled processes Ω.

2. The solution of the Cauchy problem

$$x^{\cdot} = f(x,t) + B(x)\,u^0(x,t), \qquad x(0) = x_0 \tag{1.7}$$

$$u_j^0(x,t) = -k_j^2 \langle \nabla_x w(x,t),\ b_j(x)\rangle, \qquad j = 1,\ldots,m \tag{1.8}$$

exists and satisfies the boundary condition (1.3), i.e. $\{x^0(t),\ u^0(x,t)\}\in\Omega$.

Then the least necessary energy loss J_* satisfies the inequality

$$J_* \leqslant w(x_0,0) - w(x_T,T) \tag{1.9}$$

Here ∂_x is the gradient operator with respect to the variables $x_1,\ldots,x_n$, $b_j(x)$ is the jth column vector of the matrix $B(x)$, and $\langle,\rangle$ is the scalar vector product.

To prove inequality (1.9) we consider a function $w(x, t)$ satisfying the conditions of Theorem 1. We compute the total derivative with respect to time of the function $w(x,t)$. From Eqs (1.1) we obtain

$$dw(x,t)/dt = \partial w(x,t)/\partial t + \langle \nabla_x w(x,t),\ f(x,t) + B(x)\,u\rangle \tag{1.10}$$

Because the function $w(x, t)$ is a first integral of Eqs (1.5)

$$\partial w(x,t)/\partial t + \langle \nabla_x w(x,t), f(x,t) \rangle = 0$$

Using this relation and integrating both sides of Eq. (1.10) over time, we obtain

$$w(x_T, T) = w(x_0, 0) + \int_0^T \langle \nabla_x w(x,t), B(x) u \rangle dt \qquad (1.11)$$

We consider the auxiliary functional

$$J_b[x(\cdot), u(\cdot)] = w(x_T, T) + ½G[x(\cdot)] + J[x(\cdot), u(\cdot)] \qquad (1.12)$$

By the conditions of Theorem 1, functional (1.12) is defined on the entire set of admissible controlled processes Ω.

Using formula (1.11) and elementary transformations this functional can be written in the form

$$J_b[x(\cdot), u(\cdot)] = w(x_0, 0) + \frac{1}{2}\int_0^T \sum_{j=1}^m \left[\frac{u_j(x,t)}{k_j} + k_j\langle \nabla_x w, b_j \rangle\right]^2 dt$$

from which the relation

$$\min_{\{x,u\}\in\Omega} J_b[x(\cdot), u(\cdot)] = J_b[x^0(\cdot), u^0(\cdot)] = w(x_0, 0) \qquad (1.13)$$

follows directly, where the controlled process $\{x^0(t), u^0(x,t)\}$ is determined by the solution of the Cauchy problem (1.7), (1.8).

Because the functional (1.6) is non-negative on the set Ω, the inequality

$$J_* \leqslant \min_{\{x,u\}\in\Omega} \{J[x(\cdot), u(\cdot)] + ½G[x(\cdot)]\} \qquad (1.14)$$

holds.

Using the fact that $w(x_T, T)$ does not depend on the choice of the controlled process, from relations (1.12)–(1.14) we directly obtain inequality (1.9).

We shall describe one of the possible ways of applying Theorem 1 in practice.

We know that Eq. (1.5) has n independent first integrals. Suppose that in this or some other way we have determined the independent first integrals $v_1(x,t), \ldots, v_k(x,t)$, $k \leqslant n$.

It is obvious that the function

$$w(x,t) = \sum_{i=1}^n a_i \varphi_i(v_1, \ldots, v_k) \qquad (1.15)$$

where the a_i are arbitrary parameters and the φ_i are arbitrary continuously-differentiable functions of k arguments, is also a first integral of Eqs (1.5). Under given conditions one can choose parameters a_i and functions $\varphi_i(y_1, \ldots, y_k)$ in such a way that for a $w(x,t)$ of the form (1.15) the conditions of Theorem 1 are satisfied. This determines a suitable first integral, and an estimate for J_* is obtained from formula (1.9).

Example 1. Consider the problem of estimating the least necessary energy loss to stop the rotation of an asymmetric rigid body.

The equations of the controlled rotation of a rigid body about a fixed point (centre of mass) have the form [12]

$$I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \beta_1 u_1 \quad (1\ 2\ 3) \qquad (1.16)$$

The letters in parentheses denote cyclic permutations of the indices, the I_i are the principal (central) moments of inertia, the ω_i are components of the angular velocity of the body, the $\beta_i u_i$ are controlling moments about each of the comoving axes, $\beta_i = \text{const}$, and the u_i are the controls.

We pose the problem of estimating the least value of the functional (1.4) on the set of controlled processes

$\{\omega(t), u(t)\}$ which take the dynamical system (1.6) from a given initial state $\omega_i(0) = \omega_{0i}$ to the origin of coordinates $\omega_i(T) = 1$, $i = 1, 2, 3$ in a given time T.

In the variables $x_i = I_i\omega_i b_i^{-1}$ Eqs (1.16) reduce to the form

$$x_1^{\cdot} = (I_2 - I_3)I_2^{-1}I_3^{-1}\beta_2\beta_3\beta_1^{-1}x_2x_3 + u_1 \tag{1.17}$$
$$x_{01} \doteq I_1\omega_{01}\beta_1^{-1} \qquad (1\ 2\ 3)$$

We know [12] that when the conditions

$$I_1(I_3 - I_2)\beta_2^2\beta_3^2 + I_2(I_1 - I_3)\beta_3^2\beta_1^2 + (I_2 - I_1)I_3\beta_1^2\beta_2^2 = 0 \tag{1.18}$$
$$u_1 = u_2 = u_3 = 0$$

are satisfied, Eqs (1.17) have a first integral which is the Euclidean norm of the phase vector x, i.e. the function $v(x) = (x_1^2 + \ldots + x_3^2)^{1/2}$. It is assumed below that conditions (1.18) are satisfied.

Following (1.15), we shall look for a function satisfying the conditions of Theorem 1, of the form

$$w(x) = Cv(x) \tag{1.19}$$

where C is a constant to be determined. In functional (1.4) we take $m = 3$, $k_1 = k_2 = k_3 = k$. Then, choosing controls u_i according to formulae (1.8) and (1.9), and taking Eqs (1.17) into account, we obtain

$$dv(x)/dt = -Ck^2$$

from which we deduce that for $C = v(x_0)/(k^2T)$ a function of the form (1.19) satisfies all the conditions of Theorem 1, from which we obtain the required estimate

$$J_* \leqslant v^2(x_0)/(k^2T) \tag{1.20}$$

2. An assertion with its own independent significance follows directly from Theorem 1. We shall formulate it as follows.

Theorem 2. Suppose $w(x, t)$ is a first integral of the free equations of motion (1.5), satisfying condition 1 of Theorem 1, and $x_w(T)$ is the value of the solution of the Cauchy problem (1.7), (1.8) at time T. Then the least necessary energy loss J_* for taking dynamical system (1.1) in a time T from an arbitrarily specified initial state (1.2) into a final phase state $x_T = x_w(T)$ satisfies inequality (1.9) at $x_T = x_w(T)$.

This theorem enables us to obtain estimates of the form (1.9) for all final states (1.3) lying on phase trajectories of the Cauchy problem (1.7), (1.8) defined by a first integral $w(x, t)$. Because there is considerable freedom in the choice of the functions $w(x, t)$, the estimate of the limiting possibilities of energetically optimal control systems can be obtained over a wide phase space domain, defined by the collection of points $x_w(T)$.

Example 2. Consider the problem of estimating the limiting possibilities of an energetically optimal control system for the motion of an artificial satellite (AS) in a Newtonian gravitational field about a central point O.

We place the origin of a Cartesian system of coordinates x_1, x_2, x_3 at O, and suppose that the AS is in addition acted upon by a rocket motor. Then in dimensionless variables the differential equations of the AS motion have the form [14]

$$x_1^{\cdot} = x_4, \quad x_2^{\cdot} = x_5, \quad x_3^{\cdot} = x_6$$
$$x_4^{\cdot} = u_1 - \frac{x_1}{r^3}, \quad x_5^{\cdot} = u_2 - \frac{x_2}{r^3}, \quad x_6^{\cdot} = u_3 - \frac{x_3}{r^3} \tag{2.1}$$
$$r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

Here we have used the following relations between the dimensionless coordinates x_i, the time t, the rocket forces u_i and the corresponding dimensional quantities x_{iR}, t_R, u_{iR}: $x_i = x_{iR}/r_0$, $t = t_R/(r_0/g_0)^{1/2}$, $u_i = u_{iR}/g_0$, where r_0 is a fixed (perhaps initial) distance from the centre of attraction and g_0 is the acceleration due to gravity at a distance r_0 from the centre of attraction.

The system of equations (2.1) has been the subject of numerous investigations. For example, results have been presented [14] of investigations into the behaviour of Eqs (2.1) in the case when the rocket acceleration u is constant in magnitude and direction.

It is known that differential equations for Keplerian motion (Eq. (2.1) for $u_i = 0$, $i = 1, 2, 3$) have first integrals. We will take the first integral $w(x, t)$, employed in Theorems 1 and 2 to estimate the limiting possibilities of energetically optimal control systems, to be a function of the form

$$w(x,t) = \alpha[(x_4^2 + x_5^2 + x_6^2)/2 - 1/r] \tag{2.2}$$

i.e. a function equal to the product of an arbitrary constant α and the energy integral for Keplerian motion.

We shall consider AS motion under the control specified by formulae (1.8) and (2.2). To describe this motion we obtain from (2.1), (1.8) and (2.2) the relations

$$\begin{gathered}\dot{x}_1 = x_4, \quad \dot{x}_2 = x_5, \quad \dot{x}_3 = x_6 \\ \dot{x}_4 = -\frac{x_1}{r^3} - \alpha k_1^2 x_4, \quad \dot{x}_5 = -\frac{x_2}{r^3} - \alpha k_2^2 x_5 \\ \dot{x}_6 = -\frac{x_3}{r^3} - \alpha k_3^2 x_6\end{gathered} \tag{2.3}$$

Equations (2.3) with $\alpha > 0$ describe AS motion in a centrally attracting Newtonian gravitational field, perturbed by resistive forces proportional to the projections of the AS velocity onto the axes of the Cartesian system of coordinates.

In (1.4) and (2.3) we take $m = 3$, $k_i = k$, $i = 1, 2, 3$. Then omitting the intermediate transformations, we can write down the following first integrals of the non-linear system of equations (2.3)

$$\begin{gathered}x_2 x_6 - x_3 x_5 = C_1 \exp(-\alpha k^2 t) \\ C_1 = x_{02} x_{06} - x_{03} x_{05} \qquad (1\ 2\ 3\ ,\ 4\ 5\ 6)\end{gathered} \tag{2.4}$$

where the C_i are constants determined by the initial state (1.2).

Multiplying each of the integrals (2.4) by x_1, x_2 and x_3, respectively, and adding term-by-term, we obtain

$$C_1 x_1 + C_2 x_2 + C_3 x_3 = 0 \tag{2.5}$$

Thus the AS motion in a centrally attracting Newtonian gravitational field under control given by Formulae (1.8) and (2.2) proceeds, as in the Keplerian case, in an invariant Laplace plane (2.5). Here we have the first integrals (2.4), generalizing three known scalar area integrals of Keplerian motion [14].

We denote by X_T the domain of phase space consisting of those points with coordinates equal to the value of the solution of the Cauchy problem (2.3), (1.2) at time $t = T$ and satisfying relations (2.4). An arbitrary point of the set X_T for given x_0, k is uniquely defined by the two parameters α, T.

Suppose that for the dynamical system (2.1) a final state (1.3) is chosen in the domain X_T. Then all the conditions of Theorem 2 are satisfied by the function $w(x, t)$ of form (2.2). Consequently the least necessary energy loss J_* for the displacement of the AS in a centrally attracting Newtonian gravitional field from the state (1.2) to the state $x_T \in X_T$ satisfies inequality (1.9), where the first integral $w(x, t)$ is given by formula (2.2).

As follows from relations (2.4), for the motion of an AS under the action of a control of form (1.8), (2.2), the modulus of the kinetic momentum $L(t)$ of the AS varies as follows: $L(t) = L(0)\exp(-\alpha k^2 t)$. From the formula it is clear that using relations (1.9), (2.2) one can estimate the least necessary energy loss to take the AS into the final state $x_T \in X_T$, corresponding to both increasing the kinetic momentum $L(T)$ (the case $\alpha < 0$) and to decreasing it ($\alpha > 0$) compared with the initial value $L(0)$. In the case $\alpha = 1$ the least necessary energy loss J_* does not exceed the difference between the value of the total mechanical energy of the AS in the initial phase state x_0 and final $x_T \in X_T$ phase state.

3. It is important to find all first integrals $w(x, t)$ of the free equations of motion (1.5) satisfying the conditions of Theorem 1. Denoting the set of all such first integrals by $W = \{w(x, t)\}$, estimate (1.9) can be improved and represented in the form

$$J_* \leqslant \min_{w \in W} \{w(x_0, 0) - w(x_T, T)\} \tag{3.1}$$

Furthermore, in a number of cases with the help of first integrals of the free equations of motion one can compute exactly the least value of functional (1.4) on the set of admissible controlled processes Ω.

We have the following theorem.

Theorem 3. Suppose $w(x-t)$ is a first integral of the free equations of motion (1.5), satisfying the conditions of Theorem 1 and such that the identities

$$\nabla_x \left(\sum_{j=1}^{m} k_j^2 \langle \nabla_x w(x,t),\ b_j(x) \rangle^2 \right) = 0, \quad t \in [0,T] \tag{3.2}$$

are satisfied.

Then we have the relation

$$J_* = ½[w(x_0, 0) - w(x_T, T)] \tag{3.3}$$

To prove Theorem 3 consider the auxiliary functional (1.12), where $w(x,t)$ is a specified function satisfying the conditions of Theorem 1 and identity (3.2). Then one can write down the relation

$$\min_{\{x,u\} \in \Omega} J_b[x(\cdot), u(\cdot)] = \min_{\{x,u\} \in \Omega} J[x(\cdot), u(\cdot)] + \tag{3.4}$$

$$+ w(x_T, T) + ½G[x(\cdot)]$$

It follows from this that for any first integral $w(x,t)$ satisfying the conditions of Theorem 3, the extremals of the functionals $J_b[x(\cdot),\ u(\cdot)]$ and $J[x(\cdot),\ u(\cdot)]$ on the set of admissible controlled processes Ω coincide. Here for the optimal controlled process we have $u_*(x,t) = u^0(x,t)$ and $x_*(t) = x^0(t)$, where $u^0(x,t)$, $x^0(t)$ are given by formulae (1.7) and (1.8).

Taking (1.8) into account, relation (3.4) can be represented in the form

$$\min_{\{x,u\} \in \Omega} J_b[x(\cdot), u(\cdot)] = w(x_T, T) + 2J[x_*(\cdot), u_*(\cdot)] \tag{3.5}$$

Equality (3.3) follows directly from formulae (1.13) and (3.5). Theorem 3 is proved.

Example 3. Suppose that a controlled motion of a dynamical system is described by the linear non-stationary equations

$$x^{\cdot} = A(t)x + B(t)u \tag{3.6}$$

where $A(t)$ and $B(t)$ are matrices of dimensions $(n \times m)$ and $(n \times m)$, respectively. It is required to compute the least possible value of functional (1.4) to take system (3.6) in a given time T from an arbitrarily specified initial phase state (1.2) into the final state (1.3).

We denote by $v_1(x,t), \ldots, v_n(x,t)$ the independent first integrals of free motion $x^{\bullet} = A(t)x$ that are linear with respect to phase coordinates. We consider the function

$$w(x,t) = \sum_{i=1}^{n} C_i v_i(x,t) \tag{3.7}$$

where the C_i are constants which are chosen so that the solution of the Cauchy problem of the form

$$x^{\cdot} = A(t)x + B(t)u^0(t), \quad x(0) = x_0 \tag{3.8}$$

$$u_j^0(t) = -k_j^2 \langle \nabla_x \sum_{i=1}^{n} C_i v_i(x,t),\ b_j(t) \rangle, \quad j = 1, \ldots, m$$

satisfies the boundary condition (1.3).

We know [6, 15] that under given conditions on the matrices $A(t)$ and $B(t)$ and an arbitrary final state x_T it is always possible to find constants $C_1^*, \ldots, C_n^*$ ensuring the existence of the solution of the boundary-value problem (3.8), (1.3).

We can verify directly that the function (3.7) with $C_i = C_i^*$ $(i = 1, \ldots, n)$ is a first integral of the free equations of motion and satisfies all the conditions of Theorem 3. Consequently, we have the relation

$$J_* = \frac{1}{2} \sum_{i=1}^{n} C_i^* [v_i(x_0, 0) - v_i(x_T, T)]$$

Example 4. We consider a dynamical system of the form

$$x^{\cdot} = f(x,t) + u, \quad \dim x = \dim u = n \times 1 \tag{3.9}$$

We assume that the free equations of motion (Eqs (3.9) with $u \equiv 0$) have a first integral—the Euclidean norm of the phase vector $|x| = (x_1^2 + \ldots + x_n^2)^{1/2}$. Such dynamical systems are called invariant norm systems [5, 12].

We shall show that with the help of Theorem 3 one can calculate the least necessary energy loss to take a dynamical system with invariant norm from an arbitrary initial phase state (1.2) to the origin of coordinates $x(T) = 0$.

One can verify that for $k_j = k$ ($j = 1, \ldots, n$) the function

$$w(x,t) = \alpha |x| \tag{3.10}$$

where α is an arbitrary constant, satisfies condition 1 of Theorem 1 and identities (3.2).

We will determine the constant α so that the solution of the Cauchy problem (3.9), (1.2), (1.8), (3.10) at $t = T$ is zero. Choosing the control according to formulae (1.8) and (3.10), and taking into account that $|x|$ is a first integral of Eqs (1.5), one can write down the relation $|x|^{\cdot} = -\alpha k^2$. Integrating it with respect to time between the limits $t = 0$ and $t = T$ and putting $|x(T)| = 0$, we obtain

$$\alpha = |x_0|/(k^2 T) \tag{3.11}$$

The function defined by formulae (3.10) and (3.11) satisfies all the conditions of Theorem 3. Using (3.3) the least necessary energy loss in taking the dynamical system (3.9) from the initial state (1.2) to the origin of coordinates has the form

$$J_* = |x_0|^2/(2k^2 T) \tag{3.12}$$

In conclusion we make the following remarks. Since, when conditions (1.18) are satisfied, the dynamical system (1.17) has an invariant norm, from the results of Example 4 an estimate of the form (1.20) can be replaced by the exact value of J_*, given by formulae (3.12). Comparison of the results of Examples 1 and 4 shows that in the general case it is desirable to find the set of all first integrals $w(x, t)$ satisfying the conditions of Theorem 1, and to obtain an estimate for J_* in the form of relation (3.1).

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